Characterization of Decoherence from an Environmental Perspective

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For the case of phase damping (pure decoherence) we investigate the extent to which environmental traits are imprinted on an open quantum system. The dynamics is described using the quantum channel approach. We study what the knowledge of the channel may reveal about the nature of its underlying dynamics and, conversely, what the dynamics tells us about how to consistently model the environment. We find that for a Markov phase-damping channel, that is, a channel compatible with a time-continuous Markovian evolution, the environment may adequately be represented by a mixture of only a few coherent states. For arbitrary Hilbert space dimension $N \geq 4$ we refine the idea of *quantum phase damping*, of which we show a means of identification. Symmetry considerations are used to identify decoherence-free subspaces of the system.

I. INTRODUCTION

Decoherence describes the loss of characteristic traits of quantum theory. For the success of emerging quantum technologies a detailed understanding of decoherence is of great relevance. Schemes to avoid and counter its effects need to be developed. Besides, decoherence offers insight into the much-debated quantum-to-classical transition [1–3]. The microscopic dynamics leading to decoherence might be based on very diverse grounds, reaching from purely classical phase kicks to a quantum mechanical formulation based on coupling the system of interest to some quantum environment. Hence, a further characterization of different microscopic mechanisms leading to decoherence is desirable.

Phase damping (or dephasing) denotes the case of pure decoherence, corroding the coherences of a quantum state while leaving the probabilities, that is, the diagonal elements of the density matrix, intact. The dissipation-less transition of a pure state into a classical mixture when described in the basis of energy eigenstates may serve as an example. Despite its simple nature, phase damping is enough to completely disentangle quantum states [4].

For weak system-environment coupling and short environmental correlation times decoherence may be modelled in terms of Markovian dynamics [5]. Here, the future evolution depends solely on the system's present state, rather than on anterior times. Yet, there are of course instances where this approximation is not valid. Given the dynamics of a quantum system it would be valuable to have a means of deciding whether the dynamics is Markovian or not. This point has been studied lately both in a continuous approach based on the information flow between system and environment [6], as well as from a snapshot point of view [7, 8], where the contin-uous dynamics is by construction unavailable. Rather, the state of the quantum system is known at separate times, only.

Another interesting question is whether the phase damping is due to coupling to a "real" quantum-mechanical environment, or wether it can equally be explained in terms of stochastically fluctuating, classical fields [9, 10]. The latter is a convex combination of unitary transformations, that is, random unitary (RU) dynamics. While phase damping of a single qubit or qutrit may always be described as RU dynamics, in Hilbert spaces of dimension $N \geq 4$ one cannot always find such a representation [11–13].

In the article at hand we study the characteristics of phase damping from an environmental point of view. Phase damping is described utilizing the overlap of *dynamical vectors* relative to the *phase damping basis*. The nature of the dynamics is reflected by the set of dynamical vectors, or, conversely, the properties of the dynamical vectors determine the dynamics to a certain extent. In this context, we show that in case of Markovian phase damping the dynamical vectors can be identified with coherent states. Likewise, we give instructions for a physical model of "quantum phase damping" for arbitrary Hilbert space dimension $N \geq 4$, that is, phase damping which does not allow for a RU representation.

The article is structured as follows. Section II overviews the theoretical background and serves as an introduction to the formal notation. In Sec. III we exemplary study phase damping on a single qubit, where all characteristics introduced so far actually coincide. Sections IV and V address the Markovianity and the possibility of finding a RU representation, respectively. In Sec. VI we discuss the appeerence of decoherence-free subspaces due to symmetries in our formalism.

II.   QUANTUM CHANNELS

Based on the fundamental assumption of no initial correlations between the system ϱ and its environment, the most general quantum evolution is given by a completely positive map $\mathcal{E} : \varrho \mapsto \mathcal{E}(\varrho)$. In a Hilbert space of dimension N, these maps (or "quantum channels") can always be written in terms of at most N^2 *Kraus operators* K_i such that

$$\varrho \mapsto \varrho' = \mathcal{E}(\varrho) = \sum_i K_i \varrho K_i^\dagger \qquad (1)$$

(here and in the following we denote the initial state by ϱ and its map by ϱ'). It is usually assumed that the map is trace-preserving, $\sum_i K_i^\dagger K_i = \mathbb{1}$, so as to preserve probability. If, in addition, the completely mixed state is mapped onto itself: $\sum_i K_i K_i^\dagger = \mathbb{1}$, the channel is said to be unital or doubly stochastic [14]. Throughout the article we will assume that ϱ and ϱ' live in the same Hilbert space, that is, the channel $\mathcal{E}$ maps the set of states on a Hilbert space of dimension N onto itself.

When considering a quantum channel of form (1), no particular assumptions are made about the nature of the underlying continuous dynamics. Rather, only a snapshot of the quantum system at a given time is revealed. Nevertheless, in some cases it is possible to gather information about the nature of the physical processes involved. In the remainder of this section, we want to discuss how certain additional assumptions about the structure of the channel may set restrictions to the underlying dynamics or vice versa.

Markovian Channels

A quantum channel $\mathcal{E}$ is said to be Markovian, if there exists a generator $\mathcal{L}$ of a quantum dynamical semigroup and a time $t > 0$ such that

$$\mathcal{E}(\varrho) = e^{\mathcal{L}t}\varrho \qquad (2)$$

[7, 8]. That is, the channel may be understood as a snapshot of a time-continuous Markovian evolution. The generator $\mathcal{L}$ may be written in *Lindblad* form [5]

$$\mathcal{L}(\varrho) = -i[H, \varrho] + \frac{1}{2}\sum_{i=1}^r \left\{ \left[L_i\varrho, L_i^\dagger \right] + \left[L_i, \varrho L_i^\dagger \right] \right\}. \quad (3)$$

The Markov property of a channel is closely related to the notion of *infinite divisibility* [7, 15]. A channel $\mathcal{E}$ is called infinitely divisible if, for all $\nu \in \mathbb{N}$, there exists a channel $\mathcal{E}_\nu$ with $(\mathcal{E}_\nu)^\nu = \mathcal{E}$. Surely, a Markov channel $\mathcal{E}$ is infinitely divisible: for any given $\nu \in \mathbb{N}$ it can be written as ν-fold concatenation of the channels $\mathcal{E}_\nu = e^{\mathcal{L}t/\nu}$. The converse statement, however, is not true in general [15].

RU Channels

One of the standard approaches to the quantum channel formalism is based on the reduced dynamics of a system interacting with its environment [10]. In this context, decoherence of an open quantum system is inevitably linked to growing entanglement between system and environment [2]. Yet, there are instances of irreversible dynamics that may be modeled entirely without invoking a quantum environment at all. An important example is given by RU dynamics, where the quantum channel may be written as a convex combination of unitary transformations

$$\mathcal{E}(\varrho) = \sum_i p_i U_i \varrho U_i^\dagger \qquad \left(p_i > 0, \sum_i p_i = 1 \right).$$

The dynamics may thus be thought of as originating from classical fluctuations, hence also termed "random external fields" [9, 10]. It is known, for example, that for a single qubit all doubly stochastic channels are of RU type [11]. These RU channels gain some significance in the field of quantum error correction, where they stand out due to the fact that they may be undone completely [16]. More recently they have also been applied to quantum networks [17].

Phase-Damping Channels

Phase-damping channels are among the simplest conceivable quantum channels. They are defined by the requirement that in a given basis $\{|n\rangle\}$—the *phase-damping basis*—no population transfer takes place. The only effect of the "environment" is thus to change coherences $\langle n|\varrho|m\rangle$ with $n \neq m$ and to leave all $\langle n|\varrho|n\rangle$ with $n = 1, \ldots, N$ untouched. In other words, the projectors are constants of motion: $\mathcal{E}(|n\rangle\langle n|) = |n\rangle\langle n|$ for all n.

We conclude that the Kraus operators have to be diagonal in this basis, $K_i = \mathrm{diag}(a_{i1}, a_{i2}, \ldots, a_{iN})$ and, correspondingly, the whole map $\mathcal{E}$ is diagonal, too. We find

$$\varrho'_{mn} = \langle a_n|a_m\rangle \varrho_{mn} \qquad (4)$$

with $\{a_n = (a_{1n}, a_{2n}, \ldots, a_{rn})\}$ any set of N normalized complex vectors. It is then sometimes convenient to introduce the matrix D with $D_{mn} = \langle a_n|a_m\rangle$ to write the phase-damping channel in the short form $\varrho' = D \star \varrho$, where $\star$ is the *Hadamard product*, that is, the entry-wise product of matrices of the same size: $\varrho'_{mn} = D_{mn}\varrho_{mn}$ [18]. From these considerations it is clear that phase-damping channels are among the doubly stochastic channels.

If the quantum channel $\mathcal{E}$ is defined via the system's coupling to a quantum mechanical environment, the vectors $|a_n\rangle$ may be seen as relative states of the environment, that is, relative to the states of the distinguished basis (see also Sec. V). Then the overlap $\langle a_n|a_m\rangle$, seen

as a function of time, may be related to studies of fidelity decay [19]. Yet, this relative state picture need not hold in general: the case of RU dynamics shows that in certain circumstances decoherence may be attributed to stochastic, fluctuating "classical" fields.

In many situations the dynamical vectors $|a_n\rangle$ are of course unknown a priori. In particular this holds true in an experimental setup where the matrix D is acquired via quantum process tomography [10]. One way of obtaining dynamical vectors $|a_n\rangle$ from D is by using the Cholesky factorization [20]. Given the non-negative matrix D, the Cholesky factorization gives $D = LL^\dagger$, with L a lower triangular matrix (L is in general not unique). The n-th row of L may then be identified with a complex vector $|a_n\rangle \in \mathbb{C}^d$ such that $D_{mn} = \langle a_m|a_n\rangle$. If D is a positive semi-definite matrix of rank $r < d$, there exists a unique L with columns $r + 1$ through d identical to zero [20]. That is, the vectors $|a_n\rangle$ may be chosen as elements of $\mathbb{C}^r$. In the following sections we will study what these dynamical vectors $|a_n\rangle$ reveal about the nature of the underlying dynamics.

III. THE SINGLE QUBIT CASE

Without revealing too much about the details we want to state some results of the following sections. The case of a single qubit stands out due to the fact that a phase-damping channel is always Markovian (i.e., a snapshot of Markovian dynamics) and it is of RU type. These findings of course do not allow for generalization to higher dimensional systems, yet they have some potential for building intuition. For a more rigorous approach as well as some missing definitions see Secs. IV and V.

For a single qubit the phase-damping map is defined by the matrix

$$D = \begin{pmatrix} 1 & \langle a_2|a_1\rangle \\ \langle a_1|a_2\rangle & 1 \end{pmatrix}. \tag{5}$$

Thus, a single complex number $\langle a_2|a_1\rangle =: c$ with modulus less than one determines the most general single-qubit phase-damping channel. Infinite divisibility of a phase-damping channel has to be formulated in terms of the Hadamard product (see also Sec. IV), that is, the matrix D_ν with $(D_\nu)_{mn} = (D_{mn})^{1/\nu}$ has to be checked as to its positivity. It is quite straightforward to see that the matrix D in Eq. (5) passes this test, which lets us conclude that a single qubit phase-damping channel is always Markovian (see also Sec. IV).

Another remarkable feature of single qubit phase damping—which we will later show to be intimately connected to Markovianity—is that the dynamical vectors in (4) may be chosen from the set of coherent states $\{|\alpha\rangle \,|\, \alpha \in \mathbb{C}\}$ of a harmonic oscillator. These are eigenstates of the annihilation operator, $a|\alpha\rangle = \alpha|\alpha\rangle$, and may be seen as displaced vacuum states: $|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$ [21]. In order to see this, note that for $c \neq 0$

we may simply let

$$c =: e^{-2\gamma - i\omega} \tag{6}$$

with $\gamma \in \mathbb{R}_+$ and $\omega \in [0, 2\pi)$. We then define the two-mode coherent states $|\alpha_n\rangle := e^{-i\omega_n}|\sqrt{\gamma}l_n\rangle$, where $l_1 = (1, 1), l_2 = (1, -1)$ and $\omega_1 = -\omega_2 = \omega/2$. It is easy to see that these states give the correct overlap, that is, $\langle \alpha_2|\alpha_1\rangle = c$. In this vein we can thus always define the channel in terms of coherent states $|\alpha_n\rangle$, $n = 1, 2$, leading to Markovian dynamcis. Written in Lindblad form the master equation attains its well-known form

$$\mathcal{L}(\varrho) = -i\frac{\omega}{2}[\sigma_z, \varrho] - \frac{\gamma}{2}[\sigma_z, [\sigma_z, \varrho]], \tag{7}$$

where channel (5) with c from (6) is obtained as a snapshot for $t = 1$.

Alternatively, we may choose to write the overlap of states in the form $\langle a_2|a_1\rangle = (2p - 1)e^{-i\theta}$ with $0 \leq p \leq 1$, obtaining the common quantum channel representation [10]

$$\mathcal{E}(\varrho) = e^{-i\frac{\theta}{2}\sigma_z}\left(p\varrho + (1 - p)\sigma_z\varrho\sigma_z\right)e^{i\frac{\theta}{2}\sigma_z}. \tag{8}$$

In this notation it is rather obvious, hence, that the channel is RU, which is always true for a single qubit or qutrit [11, 12].

IV. PHASE-DAMPING MARKOV PROCESSES AND COHERENT STATES

As introduced in Sec. II, the mapping of a quantum state subject to phase damping may be written using the Hadamard product. Infinite divisibility is equivalent to positivity of the matrices D_ν, where $(D_\nu)^\nu = D_\nu \star \ldots \star D_\nu = D$, $\nu \in \mathbb{N}$, i.e., $(D_\nu)_{mn} = (D_{mn})^{1/\nu}$. While it is clear that every Markov channel is infinitely divisible, note that the converse also holds in case of phase damping (when all $D_{mn} \neq 0$).

The argument is based on a theorem by Denisov [15] which states that an infinitely divisible channel $\mathcal{E}$ is of the form $\mathcal{E} = e^{\mathcal{L}}E$, where E is an idempotence with $E\mathcal{L}E = \mathcal{L}E$ [7, 15]. In case of phase damping, however, the diagonal character of the map together with the relation $E\mathcal{E} = Ee^{\mathcal{L}}E = e^{\mathcal{L}}E = \mathcal{E}$ implies $E = \mathbb{1}$ already, whenever all $D_{mn} \neq 0$. Thus, Markovianity follows directly from infinite divisibility in this case. Recall that this is certainly not true for channels in general.

From Sec. III we already know that any single-qubit phase-damping channel (with $c \neq 0$) is infinitely divisible and hence Markovian, but what about higher dimensions? A simple example shows that already for a 3-dimensional quantum system positivity may be violated: Let a 3-state phase-damping channel be given by

$$D = \begin{pmatrix} 1 & i\alpha & -i\alpha \\ -i\alpha & 1 & \alpha \\ i\alpha & \alpha & 1 \end{pmatrix},$$

with real α. Then for $\frac{1}{3} < \alpha \leq \frac{1}{2}$ the matrix D is positive, but all (Hadamard) square roots $D^{1/2}$ have one negative eigenvalue equal to $1 - \sqrt{3\alpha}$. We have thus found a single-qutrit phase-damping channel which may not be identified as a snapshot of Markovian evolution.

The notion of infinite divisibility in this section implicitly assumes all fractional powers of the initial channel to be phase damping and therefore diagonal. This excludes the rather peculiar case where some dynamics is phase damping for a particular time t only, but may well change populations at other times. Consider, for example, the unitary one-qubit map $\mathcal{E}(\varrho) = U\varrho U^{\dagger}$ with $U = \exp(-i\pi\sigma_x t)$, which is trivially phase damping for $t = 1, 2, 3, \ldots$.

From these considerations it is clear that the underlying generator $\mathcal{L}$ of the Markov dynamics is diagonal: $\mathcal{L}(|m\rangle\langle n|) = z_{mn}|m\rangle\langle n|$. This, in turn, assures the diagonal character of both the Hamiltonian $H =: \sum_n w_n |n\rangle\langle n|$ and the Lindblad operators $L_i =: \sum_n l_n^{(i)}|n\rangle\langle n|$ (see App. A), thereby leading to the relation

$$z_{mn} = -i(\omega_m - \omega_n) + \langle l_n | l_m \rangle - \frac{1}{2}(\|l_m\|^2 + \|l_n\|^2), \quad (9)$$

where $l_n := (l_n^{(1)}, \ldots, l_n^{(r)})$, r is the number of Lindblad operators in (3).

An M-mode coherent state may be written as a displacement of the vacuum

$$|\alpha\rangle = e^{-\|\alpha\|^2/2}\, e^{\alpha_1 a_1^{\dagger}} \otimes \ldots \otimes e^{\alpha_M a_M^{\dagger}}\, |0\rangle_1 \otimes \ldots \otimes |0\rangle_M$$

with bosonic creation operators $a_i^{\dagger}$ [21]. For two coherent states $|\alpha\rangle, |\beta\rangle$ this leads to an overlap

$$\langle \beta | \alpha \rangle = e^{\langle \beta, \alpha \rangle - \frac{1}{2}(\|\alpha\|^2 + \|\beta\|^2)}, \quad (10)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{C}^M$ and $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$.

Based on comparison of Eqs. (9) and (10) we define r-mode coherent states of the following form

$$|\alpha_n(t)\rangle = e^{-i\omega_n t}\left| l_n \sqrt{t} \right\rangle, \quad n = 1, \ldots, N,$$

and find that for Markovian phase damping we may define the channel in terms of *coherent states* such that

$$D_{mn} = e^{z_{mn}t}\big|_{t=1} = \langle \alpha_n(t) | \alpha_m(t) \rangle \big|_{t=1}.$$

Any Markovian phase-damping channel $\mathcal{E}$ may therefore be obtained as the reduced dynamics of the system interacting with an environment of harmonic oscillators, all in coherent states. At first sight, the time dependence of the coherent states $|\alpha_n(t)\rangle = e^{-i\omega_n t}\left|\sqrt{t} l_n\right\rangle$ may seem quite queer. Yet, this should not be too surprising given that a finite reservoir would normally lead to memory effects. In order to preserve Markovianity the dynamics thus has to be strongly driven, so as to prevent the back-flow of information from the environment to the system [6].

As a final remark we add that the $\sqrt{t}$-dependence of the centroid of the environmental coherent states reflects the fundamental relevance of standard Brownian motion for all (continuous) Markov processes.
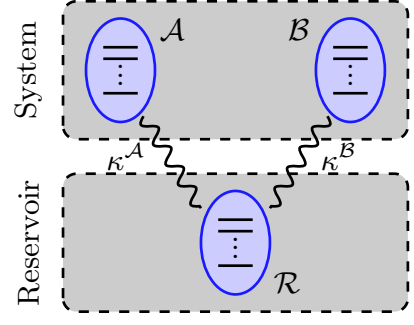

FIG. 1: (Color online) The construction of an extremal phase-damping channel is based on a bipartite system of qudits $\mathcal{A}$ and $\mathcal{B}$, locally coupling (via $\kappa^{\mathcal{A}}$ and $\kappa^{\mathcal{B}}$) to a qudit reservoir $\mathcal{R}$.

V. RU VS. QUANTUM PHASE DAMPING

We have seen in the previous section that for qutrits—or larger systems—phase-damping channels need not be Markovian. In a similar spirit, one may ask whether RU representations exists for any dimension: can all phase-damping processes be written as a convex sum of unitary maps? For a qubit, as for the question of Markovianity, the answer is positive. In general, however, the answer is no, as can be found in [11]. For a two-qubit system, that is $N = 4$, a physical model for such a non-RU (or *quantum*) phase-damping channel is described in Ref. [13].

Our aim here is to offer a method how to construct non-RU phase-damping channels in arbitrary dimension, extending earlier work. These serve as specific examples; it is an entirely different and challenging matter to test a given phase-damping channel for the RU property.

Our method of identification of such a quantum phase-damping channel rests on extremality with respect to the convex set of doubly stochastic channels. Due to a result by Landau and Streater it is known that there exist non-unitary, extremal maps in the convex set of diagonal doubly stochastic maps [11]. Extremality is guaranteed for channels where the projectors $|a_n\rangle\langle a_n|$ obtained from the dynamical vectors $\{|a_1\rangle, \ldots, |a_N\rangle\} \subset \mathbb{C}^r$ in Eq. (4) form a (possibly overcomplete) operator basis on $\mathbb{C}^r$. Note that extremality requires $r^2 \leq N$ [remember that r denotes the number of operators used in Eq. (1) or, likewise, the dimensionality of the vectors $|a_n\rangle$, N is the dimension of the quantum system].

The construction of the channel rests on a Hamiltonian H locally coupling two qudits (d-dimensional quantum systems) to a single qudit environment (cf. Fig. 1). Then, by construction, $r^2 = d^2 = N$. In the usual notation we set

$$H = H_{\mathcal{S}} + H_{\mathcal{I}} + H_{\mathcal{R}}, \quad (11)$$

where $H_{\mathcal{S}}$ and $H_{\mathcal{R}}$ denote the Hamiltonian describing system and reservoir, respectively. The local coupling of

qudits $\mathcal{A}$ and $\mathcal{B}$ to the reservoir $\mathcal{R}$ may be set to

$$H_\mathcal{I} = \sum_{i,j} \left(\kappa_{ij}^\mathcal{A} \sigma_i^\mathcal{A} \otimes \sigma_j^\mathcal{R} + \kappa_{ij}^\mathcal{B} \sigma_i^\mathcal{B} \otimes \sigma_j^\mathcal{R} \right).$$

In order to invoke a phase-damping channel on the system we have to require $H_\mathcal{S}$ as well as all operators $\sigma_i^\mathcal{A}$, $\sigma_i^\mathcal{B}$ to be diagonal (the σ-operators will be specified below).

For any given time t and assuming the usual product initial state, $\varrho \otimes \sigma$, this dynamics leads to the phase-damping channel

$$\mathcal{E}_t(\varrho) := \varrho' = \mathrm{tr}_\mathcal{R} \left(e^{-iHt} \left(\varrho \otimes \sigma \right) e^{iHt} \right)$$
$$= \mathrm{tr}_\mathcal{R} \left(U(\varrho \otimes \sigma) U^\dagger \right). \quad (12)$$

Due to the restriction to diagonal system Hamiltonian and diagonal coupling, the unitary map U allows for a diagonalization in the phase-damping basis $\{|n\rangle\}$. The interaction may thus be expressed in fashion of a controlled-unitary operation [22]

$$U = \sum_{n=1}^{d^2} |n\rangle\langle n| \otimes \tilde{U}_n. \quad (13)$$

Assuming the initial state of the reservoir to be pure, that is, $\sigma = |\psi_0\rangle\langle\psi_0|$, we obtain the phase-damping channel

$$\varrho'_{mn} = \langle \psi_n | \psi_m \rangle \varrho_{mn} \quad (14)$$

in terms of the dynamical vectors $|\psi_n\rangle := \tilde{U}_n |\psi_0\rangle$, $n = 1, \ldots, d^2$.

The properties of the phase-damping channel are now encoded in these relative environment states $|\psi_n\rangle$. In particular, the extremality of the channel is equivalent to $\{|\psi_n\rangle\langle\psi_n|\}$ being an operator basis. A constructive way of testing may be done using the Bloch representation. Recall that to a given normalized complex vector $|\psi_n\rangle \in \mathbb{C}^d$ we can assign a corresponding generalized real Bloch vector $\vec{b}_n \in \mathbb{R}^{d^2-1}$ [14]. Let $\sigma_1, \ldots, \sigma_{d^2-1}$ be orthogonal generators of $SU(d)$, that is, the σ_i are hermitian, traceless operators obeying $\mathrm{tr}\,\sigma_i\sigma_j = 2\delta_{ij}$. Together with the identity operator $\mathbb{1}$ these form an orthogonal basis of all linear operators in d dimensions, and we arrive at the Bloch representation by defining $|\psi_n\rangle\langle\psi_n| =: \vec{B}_n \cdot \vec{\sigma}$, where $\vec{B}_n = \frac{1}{2}(\frac{2}{d}, \vec{b}_n) \in \mathbb{R}^{d^2}$ and $\vec{\sigma} = (\mathbb{1}, \sigma_1, \ldots, \sigma_{r^2-1})$. For a set of d^2 projectors $\{|\psi_n\rangle\langle\psi_n|\}$ forming an operator basis, $\{\vec{B}_n\}$ is a linear independent set spanning $\mathbb{R}^{d^2}$.

We thus arrive at the following equivalence (see App. B):

The channel defined via the dynamical vectors $\{|\psi_1\rangle, \ldots, |\psi_{d^2}\rangle\}$ is an extremal channel
$$\Leftrightarrow \quad (15)$$
$$\mathrm{Vol}(\vec{b}_1, \ldots, \vec{b}_{d^2}) \neq 0.$$

We are thus able to link the extremality of the phase-damping channel to the volume $\mathrm{Vol}(\vec{b}_1, \ldots, \vec{b}_{d^2}) :=$

$1/(d^2 - 1)! \quad \det\left[(\vec{b}_2 - \vec{b}_1) \ (\vec{b}_3 - \vec{b}_1) \ \cdots \ (\vec{b}_{d^2} - \vec{b}_1) \right]$ spanned by the real vectors $\vec{b}_1, \ldots, \vec{b}_{d^2}$. In this geometric picture we can infer that the channel is extremal iff the Bloch vectors $\vec{b}_n$ do not point to the same hyperplane in $\mathbb{R}^{d^2-1}$, or, equivalently, iff the $d^2 - 1$ dimensional volume V spanned by the Bloch vectors is different from zero.

While still not a general test for the RU property, we would like to note that, nonetheless, criterion (15) may be used to give a constructive test of a channel's extremality. Given an arbitrary phase-damping channel D, the Cholesky factorization gives, as introduced in Sec. II, a set of dynamical vectors $|a_n\rangle \in \mathbb{C}^r$. Recall that r denotes the rank of the matrix D. Any r^2-dimensional subset of the corresponding Bloch vectors $\vec{b}_n$ has now to be checked for linear independence. If linear independence is found in any subset, then—following the equivalence in (15)—we may conclude upon extremality of the channel. For $r \neq 1$ this immediately excludes the RU property.

VI. SYMMETRIES AND DECOHERENCE-FREE SUBSPACES

In qubit systems it may happen that environmental influences affect different qubits in the same way. If, for instance, the wavelength of a fluctuating field is much larger than the separation of the qubits certain qubit states accumulate the same random phase and coherence among such states is preserved. To give an example consider a classical fluctuating magnetic field that couples identically to all qubits via $B(t) \sum_i \sigma_z^i =: B(t)\Sigma_z$. In such a case all superpositions of states from an eigenspace of Σ_z will not suffer from decoherence [23]. Such decoherence-free subspaces (DFS) can be identified in experiments [24].

In the quantum channel formalism the DFS appear naturally through symmetry considerations. Assume, for simplicity, an N-qubit setup where all qubits are affected by the environment in the same way. Formally, this amounts to the invariance of the channel under permutations of the qubits. In turn, the set of dynamical vectors $|a_n\rangle$ has to be invariant under qubit permutations. We conclude that $|a_n\rangle = |a_m\rangle$ whenever $\langle n| \Sigma_z |n\rangle = \langle m| \Sigma_z |m\rangle$. Thus, only $N + 1$ different dynamical vectors $|b_k\rangle$ occur with a degeneracy of $\binom{N}{k}$ (the dimension of the corresponding DFS), summing up to the total of 2^N.

To give an example, for a two-qubit system with full qubit symmetry $1 \leftrightarrow 2$ the most general phase-damping channel is made from only three dynamical vectors $|a_1\rangle = |b_1\rangle$, $|a_2\rangle = |a_3\rangle = |b_2\rangle$, $|a_4\rangle = |b_3\rangle$, such that

$$D = \begin{pmatrix} 1 & \langle b_2|b_1\rangle & \langle b_2|b_1\rangle & \langle b_3|b_1\rangle \\ \langle b_1|b_2\rangle & 1 & 1 & \langle b_3|b_2\rangle \\ \langle b_1|b_2\rangle & 1 & 1 & \langle b_3|b_2\rangle \\ \langle b_1|b_3\rangle & \langle b_2|b_3\rangle & \langle b_2|b_3\rangle & 1 \end{pmatrix} \quad (16)$$

and the space $\{|01\rangle, |10\rangle\}$ is a DFS. These considerations can of course be adapted to cases of partial symmetries of the environmental influences.

VII. CONCLUSIONS AND OUTLOOK

We study phase damping (pure decoherence) from an environmental perspective. Any given phase-damping channel may be understood in terms of an overlap of dynamical vectors $|a_n\rangle$ characterizing the channel. For a quantum environment these are relative environmental states. We investigate how the nature of a phase-damping process inflicts with properties of these dynamical vectors.

For a single qubit, we infer that any possible phase-damping channel is indeed Markovian, that is, a snapshot of some time-continuous Markovian evolution. For a single qutrit, we find examples of channels that are not Markovian: we give a class of channels we show is not infinitely divisible. Remarkably, it turns out that in case of Markovian phase damping in arbitrary dimension the dynamical vectors may be chosen to be multi-mode coherent states.

For a single qubit a phase-damping channel is of RU type. For Hilbert space dimension $N \geq 4$ we discuss a physical model of phase-damping dynamics that has no RU representation. We find that for a phase-damping channel acting on a d^2 dimensional quantum system, the RU property may be linked to a $(d^2 - 1)$-dimensional volume. In a previous article, a link between this (absolute) volume and the norm distance between the channel and the convex hull of unitary transformations was found [13].

Our considerations are of relevance for process tomography [10, 25] where it is a great challenge to reduce the dimension of the parameter space of the process. It is clear that any additional assumption about the nature of the process (phase damping, RU, Markovian) leads to further constraints. Our results allow for a characterization of the channel with a minimal number of parameters and should help to speed-up the optimization procedures involved [26].

Appendix A: Diagonal Lindblad Form

In this appendix we show that the diagonal form of the generator $\mathcal{L}$ is enough to guarantee the Hamiltonian and the Lindblad operators to be diagonal as well. In a given basis $\{|n\rangle\}$, let the generator of the semigroup $\Lambda_t = e^{\mathcal{L}t}$ be diagonal, that is, $\mathcal{L}(|m\rangle\langle n|) = z_{mn}|m\rangle\langle n|$. With $H = \sum_{mn} h_{mn}|m\rangle\langle n|$ and $L_i = \sum_{mn} l_{mn}^{(i)}|m\rangle\langle n|$ this implies

$$
\begin{aligned}
z_{mn}\delta_{rm}\delta_{ns} &= -i\left(h_{rm}\delta_{ns} - \delta_{rm}h_{ns}\right) + \frac{1}{2}\sum_i \left\{ \left(l_{rm}^{(i)}l_{ns}^{(i)} - \sum_k l_{rk}^{(i)}l_{km}^{(i)}\delta_{ns}\right) + \left(l_{rm}^{(i)}l_{ns}^{(i)} - \sum_k l_{nk}^{(i)}l_{ks}^{(i)}\delta_{rm}\right) \right\} \\
&= -i\left(h_{rm}\delta_{ns} - \delta_{rm}h_{ns}\right) + \langle l_{rm}|l_{ns}\rangle - \frac{1}{2}\sum_k \left(\langle l_{rk}|l_{km}\rangle\delta_{ns} + \langle l_{nk}|l_{ks}\rangle\delta_{rm}\right).
\end{aligned}
\tag{A1}
$$

Letting $m = n$, $r = s$, and $n \neq s$ we see that $\|l_{mr}\|^2 = 0$ for $m \neq r$, so that $l_{mr} = \delta_{mr}l_r$. Insertion into (A1) then implies $h_{mr} = \delta_{mr}\omega_r$, so that Hamiltonian and Lindblad operators are found to be diagonal. In matrix representation, the generator may thus be written as

$$
z_{mn} = -i(\omega_m - \omega_n) + \langle l_n|l_m\rangle - \frac{1}{2}(\|l_m\|^2 + \|l_n\|^2).
$$

Appendix B: Extremality Criterion

In order to see the equivalence in Eq. (15) we have to perform some matrix algebra. The vectors $|\psi_1\rangle, \ldots, |\psi_{d^2}\rangle$ define projectors giving an operator basis iff the real vectors $\vec{B}_1, \ldots, \vec{B}_{d^2}$ are linearly independent,

which is the case for [27]

$$
\begin{aligned}
\det \begin{pmatrix} \vec{B}_1 & \cdots & \vec{B}_{d^2} \end{pmatrix} &= \det \begin{pmatrix} \frac{2}{d} & \cdots & \frac{2}{d} \\ \vec{b}_1 & \cdots & \vec{b}_{d^2} \end{pmatrix} \\
&= \det \left(\begin{pmatrix} \frac{2}{d} & \cdots & \frac{2}{d} \\ \vec{b}_1 & \cdots & \vec{b}_{d^2} \end{pmatrix} \begin{pmatrix} 1 & -1 & \cdots & -1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \right) \\
&= \frac{2}{d} \det \begin{pmatrix} (\vec{b}_2 - \vec{b}_1) & (\vec{b}_3 - \vec{b}_1) & \cdots & (\vec{b}_{d^2} - \vec{b}_1) \end{pmatrix} \\
&= \frac{2(d^2 - 1)!}{d} \ \mathrm{Vol}(\vec{b}_1, \ldots, \vec{b}_{d^2}) \\
&\neq 0,
\end{aligned}
$$

where $\mathrm{Vol}(\vec{b}_1, \ldots, \vec{b}_{d^2})$ denotes the volume of the parallelogram spanned by the real, $d^2 - 1$ dimensional vectors $\{\vec{b}_1, \ldots, \vec{b}_{d^2}\}$.

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